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AKTS.
<u>Frobenius in higher algebra</u>.
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Fix a prime p.

R an $\mathbb{F}_p$-algebra.

$(x+y)^p = x^p + y^p + \cancel{p(-)}$.

$$\varphi_R : R \longrightarrow R$$
$$r \longmapsto r^p.$$

Natural in R.

$$\varphi \in \mathrm{End}_{\mathrm{End}(\mathrm{CAlg}_{\mathbb{F}_p})}(\mathrm{id})$$

$$B\mathbb{Z}_{\geq 0} \xrightarrow{\Phi} \mathrm{End}(\mathrm{CAlg}_{\mathbb{F}_p})$$
$$* \longmapsto \mathrm{id}$$
$$1 \longmapsto \varphi$$

Does this define an <u>action</u> of $B\mathbb{Z}_{\geq 0}$ on $\mathrm{Alg}_{\mathbb{F}_p}$? I.e., is Φ monoidal.

I.e., is $\mathbb{Z}_{\geq 0} \xrightarrow[\text{commutative}]{\text{monoidal}} \mathrm{End}_{\mathrm{End}(\mathrm{CAlg}_{\mathbb{F}_p})}(\mathrm{id})$?

Yes, clearly.

<u>Q</u> (Lurie, Nikolaus). Is there an analog of $B\mathbb{Z}_{\geq 0}$-action on $\mathrm{CAlg}_{\mathbb{S}}$?

<u>Remark</u>. $\exists$ coalgebra story.

X, Y spectra.

$$(X \vee Y)^{\otimes p} = X^{\otimes p} \oplus Y^{\otimes p} \oplus \text{stuff}.$$

$$\text{stuff} = C_p \otimes \text{other stuff}.$$

$$(X \vee Y)^{tC_p} = (X^{\otimes p})^{tC_p} \oplus (Y^{\otimes p})^{tC_p}.$$

In fact, $X \mapsto (X^{\otimes p})^{tC_p}$ is exact.

<u>Cor.</u> $X \xrightarrow[\text{Tate diagonal}]{\Delta} (X^{\otimes p})^{tC_p}$.

A an $\mathbb{E}_\infty$-ring.

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend right, "\varphi_A"'] & (A^{\otimes p})^{tC_p} \arrow[r] & A^{tC_p} \end{tikzcd}$$

Warning: φ is not an endomorphism.

Good news: φ exists integrally.

<u>Exs</u> 1) $A = \mathbb{S}$. $\mathbb{S} \to \mathbb{S}^{tC_p}$.

Segal conjecture: $\mathbb{S}^{tC_p} \simeq \mathbb{S}_p$.

So, $\mathbb{S} \to \mathbb{S}^{tC_p}$ is p-completion.

From now on: everything is p-completed.

So, $\varphi_{\mathbb{S}_p}$ is the identity.

2) $A = \mathbb{F}_2$. $\mathbb{F}_2 \longrightarrow \mathbb{F}_2^{tC_2} \simeq \prod_{i \in \mathbb{Z}} \mathbb{F}_2[i]$

Stable total deg. p power op.

$(\ldots, 0, 0, 1, Sq^1, Sq^2, \ldots)$.

3) R discrete commutative rings.

$$R \longrightarrow R/p$$

$$r \longmapsto r^p$$

$\pi_0 \varphi$.

4) X finite complex.

$$A = \mathbb{S}^X = F(X_+, \mathbb{S})$$

$$\mathbb{S}^X \to (\mathbb{S}^X)^{tC_p}.$$

$(-)^{tC_p}$ commutes w/ finite limits.

$X = *$ (ex 1).

So, $\mathbb{S}^X \xrightarrow{\simeq} (\mathbb{S}^X)^{tC_p}$ if X is a finite CW complex.

<u>Observe</u>. X finite spectrum, $X \xrightarrow{\simeq} X^{tC_p}$.

$\Rightarrow$ A $\mathbb{E}_\infty$ finite spectrum, $A \xrightarrow{\simeq} A^{tC_p}$.

<u>Guiding Q</u>. Is there a monoidal functor

$$B\mathbb{Z}_{\geq 0} \longrightarrow \mathrm{End}(\mathrm{CAlg}_p^{fin})$$

$$\mathbb{Z}_{\geq 0} \longrightarrow \mathrm{End}_{\mathrm{End}(\mathrm{CAlg}_p^{fin})}(\mathrm{id})$$

$\uparrow$

Map of $\mathbb{E}_2$-monoid?

V $\mathbb{F}_p$-vector space.

"Definition". There is a universal functor

$$Sp^{BV} \xrightarrow[(-)^{tV}]{} Sp$$

proper Tate, kills things induced from proper subgroups.

Two types of maps. A $\mathbb{E}_\infty$.

1) Frobenius

$$\begin{tikzcd} A \arrow[r] \arrow[dr, bend right, "\varphi^V"'] & (A^{\otimes V})^{tV} \arrow[d] \\ & A^{tV} \end{tikzcd}$$

$U \hookrightarrow V$, $\varphi^V_U : A^{tU} \to A^{tV}$.

2) Canonical.

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend right, "\mathrm{can}^V"'] & A^{hV} \arrow[r] & A^{tV} \end{tikzcd}$$

$V \twoheadrightarrow W$.

can^V_W $\quad A^{tW} \to A^{tV}$.

Recall. Quillen's Q-construction. $Q\mathrm{Vect}^{f.d.}_{\mathbb{F}_p}$

Obj: $\mathbb{F}_p$ f.d.v.s. V

Mor:

$$\begin{tikzcd} & W \arrow[dl, hookleftarrow] \arrow[dr, twoheadrightarrow] & \\ U & & V \end{tikzcd}$$

$Q\mathrm{Vect}^{f.d.}_{\mathbb{F}_p}$ is symmetric monoidal via $\oplus$.

Thm (Y). There is a canonical oplax monoidal functor

$$\mathrm{QVect}_{\mathbb{F}_p} \longrightarrow \mathrm{End}(\mathrm{CAlg})$$

$$V \longmapsto (-)^{\tau V}$$

$$\begin{tikzcd}[row sep=small, column sep=small] & U \arrow[dl, equal] \arrow[dr] & \\ U & & V \end{tikzcd} \longmapsto \varphi^V_U : (-)^{\tau U} \to (-)^{\tau V}.$$

$$\begin{tikzcd}[row sep=small, column sep=small] & V \arrow[dl] \arrow[dr, equal] & \\ W & & V \end{tikzcd} \longmapsto \mathrm{can}^V_W : (-)^{\tau W} \to (-)^{\tau V}.$$

$$\left((-)^{\tau V}\right)^{\tau V} \longleftarrow (-)^{\tau(U \oplus V)}.$$

Rems. 1) Don't need to p-complete.

2) Works for all finite ab. groups.

3) On π_*, composition of power ops.

Def. An $\mathbb{E}_\infty$-ring A is Frobenius stable if A is p-complete and $\mathrm{can}^V : A \simeq A^{\tau V}$.

$\mathrm{CAlg}^F_p \subseteq \mathrm{CAlg}$.

$\cup$

$\mathrm{CAlg}^{\mathrm{perf}}_p$ those where φ^V are $\simeq$s.

Cor. There is a canonical action of S^1 on $\mathrm{CAlg}^{\mathrm{perf}}_p$ via Frobenius.

$$\begin{tikzcd} \mathrm{QVect}_{\mathbb{F}_p} \arrow[r, "\text{monoidal}"] \arrow[d, dash] & \mathrm{End}(\mathrm{CAlg}^{\mathrm{perf}}_p) \\ S^1 \simeq B\mathbb{Z} \simeq_p |\mathrm{QVect}_{\mathbb{F}_p}| \arrow[ur, dashed] & \end{tikzcd}$$

<u>Def</u>. $L \subseteq \mathrm{Mor}(\mathrm{QVect}^{f.d}_{\mathbb{F}_p})$ left morphisms.

$$\begin{tikzcd} & V \arrow[dl] \arrow[d, equal] \\ W & V. \end{tikzcd}$$

$$K^{part}(\mathbb{F}_p) = \mathrm{End}_{\mathrm{QVect}_{\mathbb{F}_p}[L^{-1}]}(*).$$

$$\begin{tikzcd} \mathrm{QVect}_{\mathbb{F}_p} \arrow[r] \arrow[d] & \mathrm{End}(\mathrm{CAlg}^F_p) \\ \mathrm{QVect}_{\mathbb{F}_p}[L^{-1}] \arrow[ur, dashed, "\text{Exists by def.}"'] & \end{tikzcd}$$

<u>Thm</u> (?). $K^{part}(\mathbb{F}_p) \xrightarrow{\simeq} \pi_0 K^{part}(\mathbb{F}_p) \simeq \mathbb{Z}_{\geq 0}$.

<u>Cor</u>. $K^{part}(\mathbb{F}_p) \simeq_p B\mathbb{Z}_{\geq 0}$.

<u>Cor</u>. Natural <u>action</u> of $B\mathbb{Z}_{\geq 0}$ on CAlg^F_p via Frobenius.

<u>Application</u> (Mandell, Nikolaus).

$\mathcal{S}^{fin}_p$ = p-complete nilpotent finite spaces.

$$\mathcal{S}^{fin}_p \longleftarrow (\mathrm{CAlg}^{perf}_p)^{hS^1}$$

$$X \longmapsto \mathbb{F}^X.$$